

CONTINUA IN WHICH ALL CONNECTED SUBSETS ARE ARCWISE CONNECTED

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ABSTRACT. Let X be a metric continuum such that every connected subset of X is arcwise connected. Some facts concerning the distribution of local cutpoints of X are obtained. These results are used to prove that X is a regular curve.

1. Introduction. Several attempts have been made to characterize the spaces in which all connected subsets are arcwise connected, e.g. Kuratowski and Knaster [1], Whyburn [3], [5], [6] and Tymchatyn [2]. In [10] Mohler conjectured that if X is a metric continuum such that each connected subset of X is arcwise connected, then X is a regular curve. The main purpose of this paper is to resolve this conjecture in the affirmative.

The reader may consult Whyburn [8, V. 2] for a survey of the properties of hereditarily locally connected continua and [8, III. 9] for a treatment of local cutpoints. Our notation follows Whyburn [8]. We collect here some basic definitions for the convenience of the reader. A continuum is a nondegenerate, compact, connected metric space. A continuum is said to be *hereditarily locally connected* if each of its subcontinua is locally connected. A continuum is said to be *regular* if it has a basis of open sets with finite boundaries. An *arc* is a homeomorph of the closed unit interval $[0, 1]$. If A is an arc and $c, d \in A$ then $[c, d]$ denotes the arc in A with endpoints c and d . A subset A of a space X is said to be *arcwise connected* if every pair of points of A can be joined by an arc in A . A point p in a continuum X is said to be a *local cutpoint* or *local separating point* of X if there is a connected open set U in X such that $U - \{p\}$ is not connected. The term *neighbourhood* will always mean open neighbourhood. We denote the closure of a set A by $\text{Cl}(A)$.

I wish to thank Professor A. Lelek and Professor L. Mohler for several helpful communications regarding this problem.

Received by the editors July 24, 1973 and, in revised form, February 14, 1974.

AMS (MOS) subject classifications (1970). Primary 54F50; Secondary 54F20, 54F25.

Key words and phrases. Regular curves, arcwise connected, continua, hereditarily locally connected.

(¹) This research was supported in part by a grant from the National Research Council of Canada.

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2. In this section, we study several conditions that are satisfied by continua whose connected subsets are arcwise connected. The following theorem has played a central role in the study of these continua.

THEOREM 1 (Tymchatyn [2]). *Let X be a continuum such that $X = A_0 \cup A_1 \cup \dots$ where*

- (i) A_0 is not empty and contains no local cutpoints of X ,
- (ii) for each $i = 1, 2, \dots$, A_i is a closed set,
- (iii) for each $i \neq j$, $A_i \cap A_j$ is void.

Then X contains a connected set that is not arcwise connected.

LEMMA 2. *If X is a locally connected continuum such that L , the set of local cutpoints of X , is totally disconnected then L can be written as the union of a countable family of pairwise disjoint closed sets.*

PROOF. It is well known (see [8, p. 63]) that L is an F_σ . Since L is totally disconnected it is easy to see that L can be written as the union of a countable family of pairwise disjoint closed sets.

By a *null sequence* of sets is meant a sequence of sets whose diameters converge to zero.

THEOREM 3. *Let X be a continuum such that every sequence of disjoint subcontinua of X is a null sequence. Then the following are equivalent:*

(a) *If $A_1, A_2, \dots$ is any sequence of pairwise disjoint closed subsets of X and x and y are points that are separated by $X - (A_1 \cup A_2 \cup \dots)$ then some countable subset of $X - (A_1 \cup A_2 \cup \dots)$ separates x and y in X .*

(b) *If A is any subcontinuum of X and $x, y \in A$ then there is an arc in A which contains x and y and which contains at most countably many points that are not local cutpoints of A .*

(c) *If A is any subcontinuum of X then the set of local cutpoints of A is not contained in the union of countably many pairwise disjoint closed proper subsets of A .*

PROOF. (a) $\Rightarrow$ (c). Suppose (c) fails. Then there is a subcontinuum A of X such that the set of local cutpoints of A is contained in the union of a countable family $A_1, A_2, \dots$ of closed, proper, pairwise disjoint subsets of A . We may suppose without loss of generality that A_1 and A_2 are nonempty sets. Let $x \in A_1$ and let $y \in A_2$. Since every sequence of pairwise disjoint subcontinua of X is a null sequence we may suppose that the A_i form a null sequence. It follows that the decomposition space Y obtained from A by identifying each of the sets A_i to a point is a compact metric space. The image of the set $A_1 \cup A_2 \cup \dots$

in Y under the natural projection π is a countable set. Hence $A - (A_1 \cup A_2 \cup \dots)$ separates x and y in A since $Y - \pi(A_1 \cup A_2 \cup \dots)$ separates $\pi(x)$ from $\pi(y)$ in Y . By [8, III. 9.41] every set in $A - (A_1 \cup A_2 \cup \dots)$ which separates x and y is uncountable. In particular every set in X which separates x and y is uncountable. Thus, (a) also fails.

(b) $\Rightarrow$ (c). Suppose (c) fails. Then there is a subcontinuum A of X such that the set of local cutpoints of A is contained in the union of a countable family $A_1, A_2, \dots$ of closed proper subsets of A . By Sierpinski's theorem there exist $x, y \in A - (A_1 \cup A_2 \cup \dots)$ and each arc in A which contains x and y contains uncountably many points of $A - (A_1 \cup A_2 \cup \dots)$. Hence, every arc in A which contains x and y contains uncountably many points that are not local cutpoints of A . Thus, (b) also fails.

(c) $\Rightarrow$ (b). Suppose (b) fails. Let L denote the set of local cutpoints of X . We may suppose without loss of generality that there exist $x, y \in X$ such that each arc in X which contains both x and y contains uncountably many points of $X - L$. For each $z \in X$ let $A(z) = \bigcup \{A \subset X \mid A \text{ is a continuum, } z \in A \text{ and } A - L \text{ is countable}\}$.

By [7, Theorem 34] each $A(z)$ is closed. Clearly, $A(z)$ is also connected. Define an equivalence relation $\sim$ on X by setting $x \sim y$ if and only if $x \in A(y)$. Since the set of nondegenerate equivalence classes of $\sim$ is a null sequence, it follows that $\sim$ is a closed relation.

Let π be the natural projection of X onto the quotient space $X/\sim$. X is hereditarily locally connected since it can contain no continuum of convergence. Hence, $X/\sim$ is also a Peano continuum. A point $z \in X/\sim$ is a local cutpoint of $X/\sim$ only if $\pi^{-1}(z)$ contains a local cutpoint of X . Also, if p is a local cutpoint of X and $A(p) = \{p\}$ then $\pi(p)$ is a local cutpoint of $X/\sim$. We shall prove that the set of local cutpoints of $X/\sim$ is totally disconnected. By Lemma 2 it will follow that the set of local cutpoints of $X/\sim$ is the union of a countable family of pairwise disjoint closed sets and hence the set of local cutpoints of X is contained in the union of a countable family of pairwise disjoint closed sets. Thus, it will have proved that (c) also fails.

Just suppose that the set of local cutpoints of $X/\sim$ is not totally disconnected. Since $X/\sim$ is a Peano continuum the set of local cutpoints of $X/\sim$ is an F_σ . It follows by the Sum Theorem for dimension zero that the set of local cutpoints of $X/\sim$ contains a continuum A . Then $\pi^{-1}(A)$ is a continuum in X which is the union of uncountably many equivalence classes of $\sim$. Let $c, d \in \pi^{-1}(A)$ such that $c \not\sim d$. Let C be an arc in $\pi^{-1}(A)$ with endpoints c and d . For convenience, we identify C with the closed unit interval $[0, 1]$ with its usual order and its usual metric. To prove the theorem, it will suffice to prove that there is

an arc $D \subset \pi^{-1}(A)$ such that $c, d \in D$ and $D - L$ is at most countable.

Let $A_1, A_2, \dots$ denote the nondegenerate equivalence classes of $\sim$. If $x \in C - L$ then $x \in A_1 \cup A_2 \cup \dots$. Let m_1 be an integer such that $\text{diameter}(A_{m_1} \cap C) \geq \text{diameter}(A_i \cap C)$ for each $i = 1, 2, \dots$. Let $c_i = \min(A_{m_1} \cap C)$ and let $d_i = \max(A_{m_1} \cap C)$. Let B_1 be an arc in A_{m_1} with endpoints c_1 and d_1 such that $B_1 - L$ is at most countable. Notice that $A_{m_1} \cap C \subset [c_1, d_1]$ and for each $i = 1, 2, \dots$ there do not exist $a, b \in A_i \cap C$ such that $a < c_1 < d_1 < b$.

Suppose $A_{m_1}, \dots, A_{m_n}$ have been selected and for each $j = 1, \dots, n$, B_j is an arc in A_{m_j} with endpoints c_j and d_j such that $B_j - L$ is at most countable and $(A_{m_j} \cap C) \subset [c_1, d_1] \cup \dots \cup [c_j, d_j]$. Suppose also that there do not exist $k \in \{1, \dots, n\}$ and an integer $i \in \{1, 2, \dots\}$ with $a, b \in (C \cap A_i) - ([c_1, d_1] \cup \dots \cup [c_n, d_n])$ such that $a < c_k < d_k < b$. Suppose the intervals $[c_j, d_j]$ are disjoint.

If, for each j , $(A_j \cap C) - ([c_1, d_1] \cup \dots \cup [c_n, d_n])$ contains at most one point, then it is easy to check that $D = (C - ([c_1, d_1] \cup \dots \cup [c_n, d_n])) \cup B_1 \cup \dots \cup B_n$ is an arc in $\pi^{-1}(A)$ which contains c and d and $D - L$ is at most countable.

Let us suppose, therefore, that there exists an integer m_{n+1} such that

$$\begin{aligned} 0 &< \text{diameter}((A_{m_{n+1}} \cap C) - ([c_1, d_1] \cup \dots \cup [c_n, d_n])) \\ &\geq \text{diameter}((A_j \cap C) - ([c_1, d_1] \cup \dots \cup [c_n, d_n])) \end{aligned}$$

for each $j = 1, 2, \dots$.

Let

$$c_{n+1} = \min((A_{m_{n+1}} \cap C) - ([c_1, d_1] \cup \dots \cup [c_n, d_n]))$$

and let

$$d_{n+1} = \max((A_{m_{n+1}} \cap C) - ([c_1, d_1] \cup \dots \cup [c_n, d_n])).$$

Let B_{n+1} be an arc in $A_{m_{n+1}}$ with endpoints c_{n+1} and d_{n+1} such that $B_{n+1} - L$ is at most countable.

Let $D = (C - ([c_1, d_1] \cup [c_2, d_2] \cup \dots)) \cup B_1 \cup B_2 \cup \dots$. For each positive integer i , $A_i \cap (D - ([c_1, d_1] \cup [c_2, d_2] \cup \dots))$ contains at most one point. Thus, $D - L$ is at most countable.

For each positive integer j let $h_j: [c_j, d_j] \rightarrow B_j$ be a homeomorphism such that $h_j(c_j) = c_j$ and $h_j(d_j) = d_j$. Define $h: C \rightarrow D$ by

$$h(x) = \begin{cases} x & \text{if } x \notin [c_j, d_j] \text{ for any } j, \\ h_j(x) & \text{if } x \in [c_j, d_j]. \end{cases}$$

Then h is easily seen to be a homeomorphism of the arc C onto D . Thus D is an arc in $\pi^{-1}(A)$ which contains c and d and $D - L$ is at most countable. Thus, $\pi(c) = \pi(d)$ which is a contradiction.

(c) $\Rightarrow$ (a). Suppose (a) fails. Then there is a sequence $A_1, A_2, \dots$ of pairwise disjoint proper closed subsets of X and points x and y that are separated by $X - (A_1 \cup A_2 \cup \dots)$ such that no countable subset of $X - (A_1 \cup A_2 \cup \dots)$ separates x and y in X . Let $A = A_1 \cup A_2 \cup \dots$. We shall prove that X contains a subcontinuum B such that $x, y \in B$, no countable subset of $B - A$ separates x and y in B and $B - A$ contains only countably many local cutpoints of B . Clearly the set of local cutpoints of B is contained in the union of countably many disjoint proper closed sets so (c) fails.

We define by transfinite induction a nest of subcontinua B_α of X as follows: Let $B_0 = X$. Let α be a countable ordinal number. Suppose that for each $n < \alpha$, B_n has been defined to be a subcontinuum of X such that $x, y \in B$ and no countable subset of $B_n - A$ separates x and y in B_n . If, for some $n < \alpha$, $B_n - A$ contains only countably many local cutpoints of B_n , we are done. Suppose, therefore, that for each $n < \alpha$, $B_n - A$ has uncountably many local cutpoints of B_n .

Case 1. α is the successor of an ordinal number m . By assumption $B_m - A$ contains uncountably many local cutpoints of B_m . It follows from [8, III. 9.21] that every uncountable set of local cutpoints of a continuum contains a pair of points that separate the continuum. Thus, there exist $a_\alpha, b_\alpha \in B_m - (A \cup \{x, y\})$ such that $B_m - \{a_\alpha, b_\alpha\}$ is not connected. Let B_α be the closure of the component of $B_m - \{a_\alpha, b_\alpha\}$ that contains x and y . Then, no countable set of $B_\alpha - A$ separates x and y in B_α .

Case 2. α is a limit ordinal. Let $B_\alpha = \bigcap_{n < \alpha} B_n$. We shall show that no countable subset of $B_\alpha - (A \cup \{x, y\})$ separates x and y in B_α . Let C be a countable subset of B_α .

Let $C' = C \cup \bigcup_{n < \alpha} \{a_{n+1}, b_{n+1}\}$. Since C' is countable x and y lie in the same component E of $X - C'$. There is an arc D in the locally connected, topologically complete, metric space E with endpoints x and y . By induction it is easy to see that $D \subset B_n$ for each $n < \alpha$. Hence $D \subset B_\alpha - C$ which implies that x and y lie in the same component of $B_\alpha - C$.

Since X does not contain uncountably many pairwise disjoint nondegenerate subcontinua it follows that for some countable ordinal α , B_α has at most countably many local cutpoints of B_α in $B_\alpha - A$. This completes the proof of Theorem 3.

Corollary III. 9.21 in Whyburn [8] asserts that if X is a metric continuum and if G is an uncountable set of local cutpoints of X then there is a countable subset G_0 of G such that every point of $G - G_0$ is of order 2 relative to $G - G_0$. In particular, there exist $a, b \in G - G_0$ such that $X - \{a, b\}$ is not connected.

It is easy to see that $X - \{a, b\}$ has either 2 or 3 components.

If X is as above and C is a connected subset of X then $\text{Cl}(C) - C$ contains at most countably many local cutpoints of $\text{Cl}(C)$. For let G be an uncountable set of local cutpoints of $\text{Cl}(C)$. Let $a, b \in G$ such that $\text{Cl}(C) - \{a, b\}$ is not connected. Since C is connected and dense in $\text{Cl}(C)$ it follows that $C - \{a, b\}$ is not connected. Thus, at least one of a and b is in C and $G \not\subset \text{Cl}(C) - C$.

In order that all of the connected subsets of a regular continuum should be arcwise connected it is necessary that the continuum satisfy conditions (a)–(b) of Theorem 3. If X is a regular continuum that satisfies condition (a) of Theorem 3, let Y be a connected set in X . Let D be a countable dense set in Y and let $a \in D$. By Theorem 3 for each $d \in D$ there is an arc A_d in $\text{Cl}(Y)$ such that $a, d \in A_d$ and A_d contains at most countably many points that are not local cutpoints of $\text{Cl}(Y)$. By the last paragraph $A_d - Y$ is at most countable. By [7, Theorem 34] $Y \cup \bigcup \{A_d | d \in D\}$ is arcwise connected. Thus, there is a countable set $C = \bigcup \{A_d - Y | d \in D\}$ such that $Y \cup C$ is arcwise connected.

Question 1. If X is a regular continuum that satisfies condition (a) of Theorem 3, is every connected subset Y of X arcwise connected?

Question 2. If X is as in Question 1, is every connected F_σ in X arcwise connected? In particular is the set of local cutpoints of X arcwise connected if it is connected?

Question 3. Let X be a regular continuum such that every pair of separated sets in X can be separated by a countable set. Is every connected subset of X arcwise connected?

THEOREM 4. *Let X be a regular continuum. If C is a connected subset of X then C cannot be decomposed into countably infinitely many pairwise disjoint sets that are closed in C .*

PROOF. Let C be a connected subset of X . Suppose $C = \bigcup_{i=1}^{\infty} A_i$ where the A_i are pairwise disjoint proper subsets of C which are closed in C . Since every sequence of pairwise disjoint connected sets in a regular space is a null sequence we may suppose the sequence A_i is a null sequence.

Define an equivalence relation $\sim$ on C by letting $x \sim y$ if and only if there is a natural number i such that $x, y \in A_i$. Let π be the natural projection of C onto $C/\sim$. Now $C/\sim$ with the quotient topology is a connected countable space. We shall obtain a contradiction by proving that $C/\sim$ cannot be connected because it is a countable, T_1 , normal space. It is clear that $C/\sim$ is a T_1 space since $\pi^{-1}(p)$ is closed in C for each $p \in C/\sim$. It remains to prove only that $C/\sim$ is normal.

Let M and N be disjoint closed sets in $C/\sim$. Then $\pi^{-1}(M)$ and $\pi^{-1}(N)$

are disjoint closed sets in C . Since C is normal there exist disjoint open sets U and V in C such that $\pi^{-1}(M) \subset U$ and $\pi^{-1}(N) \subset V$. Since the sequence A_i is a null sequence of closed sets in X it follows that $\pi^{-1}(\pi(C - U))$ is a closed set in C which misses $\pi^{-1}(M)$. Hence, $U - \pi^{-1}(\pi(C - U))$ and $V - \pi^{-1}(\pi(C - V))$ are neighbourhoods of $\pi^{-1}(M)$ and $\pi^{-1}(N)$ respectively whose images under π are disjoint neighbourhoods of M and N respectively in $C/\sim$.

COROLLARY 5. *A connected subset of a regular continuum has either one or uncountably many arc components.*

PROOF. The arc components of a connected subset of a regular continuum are closed (see [7, p. 334]); hence Theorem 4 applies.

3. Continua that are not regular. For the remainder of this section let X be a fixed continuum that is not regular but which contains no nonnull sequence of pairwise disjoint subcontinua.

DEFINITION. Let M be a subcontinuum of a continuum X and let $x, y \in M$. We say M satisfies property $P(x, y)$ if no finite set separates x and y in any neighbourhood of M .

THEOREM 6. *If X is a continuum that is not regular then X contains a connected subset that is not arcwise connected.*

PROOF. We may suppose as in [2] that X is a hereditarily locally connected continuum and that every sequence of pairwise disjoint subcontinua of X is a null sequence. By Lemma 2 we may suppose that for each subcontinuum P of X the set of local cutpoints of P is not totally disconnected.

Let $a \in X$ such that X is not regular at a . Let M denote the set of points of X which cannot be separated from a by a finite set in X . By Whyburn [8, V. 4.4, 4.5] M is a nondegenerate continuum. By Whyburn [8, III. 9.2] at most a countable number of points of M are local cutpoints of X .

The first three claims can be proved by contradiction. The proofs are straightforward and are omitted.

CLAIM 1. *M satisfies $P(a, b)$ for each $b \in M - \{a\}$.*

Let $b \in M - \{a\}$. Let $M_\lambda, \lambda \in \Lambda$ be a maximal nest of subcontinua of M each of which satisfies property $P(a, b)$ and let $P_{ab} = \bigcap_{\lambda \in \Lambda} M_\lambda$. Then P_{ab} is a continuum which is irreducible with respect to satisfying $P(a, b)$, for if U is a neighbourhood of P_{ab} , then $M_\lambda \subset U$ for some $\lambda \in \Lambda$ and hence no finite set separates a and b in U . We shall describe in great detail the structure of the continuum P_{ab} .

In [2, Theorem 6] the author considered the case where P_{ab} is an arc. To see that P_{ab} may be quite complicated let Y be the continuum constructed in the proof of Theorem 6 in [2]. Then $Y = A_0 \cup A_1 \cup A_2 \cup \dots$ where $A_1, A_2, \dots$ are pairwise disjoint arcs and A_0 is also an arc. Let Z be the decomposition space obtained from Y by contracting A_i to a point for each $i = 1, 2, \dots$. Then $Z = P_{ab}$ where a and b are the endpoints of the arc A_0 .

CLAIM 2. *If $c, d \in P_{ab}$ then P_{ab} satisfies $P(c, d)$.*

CLAIM 3. *If U is a neighbourhood of $x \in P_{ab}$ such that $b \notin U$ then there is a point c in the boundary of U and a continuum $P \subset P_{ab} - U$ such that P satisfies $P(b, c)$.*

If $x \in P_{ab}$ is not a local cutpoint of X and Q and P are continua in P_{ab} which satisfy $P(a, x)$ and $P(x, b)$ respectively then $P \cup Q$ is a continuum satisfying $P(a, b)$ and so $P \cup Q = P_{ab}$ by the irreducibility of P_{ab} .

CLAIM 4. *If $x \in P_{ab} - \{b\}$ then there exists a proper subcontinuum of P_{ab} which satisfies $P(a, x)$.*

PROOF. Suppose the claim is false. Let $x \in P_{ab} - \{b\}$ such that no proper subcontinuum of P_{ab} satisfies $P(a, x)$.

If U is any neighbourhood of b in P_{ab} then $\text{Cl}(U) - U$ contains at most countably many local cutpoints of $\text{Cl}(U)$ by the argument following Theorem 3. If U is connected, then by Lemma 2 there exists an arc I of local cutpoints of $\text{Cl}(U)$. Now I intersects the boundary of U in a set that is compact and at most countable so U contains an arc of local cutpoints of U and hence of P_{ab} .

By the last paragraph, there exists a sequence I_i of pairwise disjoint arcs in $P_{ab} - \{b\}$ such that $\limsup I_i = \{b\}$ and for each $i = 1, 2, \dots$ each point of I_i is a local cutpoint of P_{ab} .

Let c_i and d_i be the endpoints of I_i . Since P_{ab} contains only countably many local cutpoints of X we may suppose c_i and d_i are not local cutpoints of X . By the argument following Theorem 3 we may suppose that c_i and d_i are points of order 2 in P_{ab} , that c_i and d_i separate P_{ab} into either 2 or 3 components and that the component K_i of $P_{ab} - \{c_i, d_i\}$ which meets I_i contains neither a nor b . By Whyburn [3, §4] we may suppose that every point of $I_i - \{c_i, d_i\}$ disconnects K_i . Let (by [8, III. 9.21]) $z_i \in I_i - \{c_i, d_i\}$ be a point of order 2 in P_{ab} such that z_i is not a local cutpoint of X . Then z_i separates K_i into exactly two components. Finally, we may assume that the sets K_i are pairwise disjoint. For each i let U_i be a neighbourhood of K_i such that $\text{Cl}(U_i) \cap \text{Cl}(U_j)$ is empty for $i \neq j$. Notice that $\text{Cl}(K_i) = K_i \cup \{c_i, d_i\}$.

By Claim 3 there is a component P of $P_{ab} - K_i$ such that P satisfies either $P(a, c_i)$ or $P(a, d_i)$. We may suppose without loss of generality that P satisfies $P(a, c_i)$. Since c_i is not a local cutpoint of X it follows that P does not satisfy $P(c_i, b)$ for otherwise P would be a proper subcontinuum of P_{ab} which satisfies $P(a, b)$. By Claim 3 it follows that there is a component Q of $P_{ab} - K_i$ such that Q satisfies $P(d_i, b)$. Let $P_{ac_i} \subset P$ and $P_{d_i b} \subset Q$ be continua which are irreducible with respect to satisfying $P(a, c_i)$ and $P(d_i, b)$ respectively. If $P_{ac_i} \cap P_{d_i b}$ is infinite let U be any neighbourhood of $P_{ac_i} \cup P_{d_i b}$. Let A be a finite set in U and let $x \in (P_{ac_i} \cap P_{d_i b}) - A$. By Claim 2, P_{ac_i} satisfies $P(a, x)$ and $P_{d_i b}$ satisfies $P(x, b)$. Since U is a neighbourhood of both P_{ac_i} and $P_{d_i b}$, A does not separate a from x or x from b in U . Thus A does not separate a from b in U and so $P_{ac_i} \cup P_{d_i b}$ is a proper subcontinuum of P_{ab} which satisfies $P(a, b)$. With this contradiction we conclude that $P_{ac_i} \cap P_{d_i b}$ is finite. A similar argument can be used to show that every point of $P_{ac_i} \cap P_{d_i b}$ is a local cutpoint of X .

By using the facts that $P_{ab} - K_i$ does not satisfy $P(a, b)$ while P_{ab} satisfies $P(a, b)$, that $\text{Cl}(K_i) - K_i = \{c_i, d_i\}$ and that c_i and d_i are not local cutpoints of X one can easily prove by contradiction that $\text{Cl}(K_i)$ satisfies $P(c_i, d_i)$. Since c_i and d_i are not local cutpoints of X it follows from the remark following Claim 3 that $P_{ab} = P_{ac_i} \cup K_i \cup P_{d_i b}$.

Since z_i separates c_i and d_i in $\text{Cl}(K_i)$ it follows from Claim 3 that $\text{Cl}(K_i)$ satisfies $P(c_i, z_i)$. Since c_i is not a local cutpoint of X , $P_{ac_i} \cup \text{Cl}(K_i)$ satisfies $P(a, z_i)$. Let $P_{az_i} \subset P_{ac_i} \cup \text{Cl}(K_i)$ be a continuum which is irreducible with respect to satisfying $P(a, z_i)$.

Since $b \notin P_{ac_i} \cup \text{Cl}(K_i)$ we may suppose that, for each i , $z_{i+1} \in P_{d_i b}$. Hence, $\text{Cl}(K_{i+1}) \subset P_{d_i b}$. By Claim 2 we may suppose that $P_{d_{i+1} b} \subset P_{d_i b} - K_{i+1}$. It now follows that $P_{az_i} \subset P_{ac_i} \cup \text{Cl}(K_i) \subset P_{ac_{i+1}} \subset P_{az_{i+1}}$.

Since P_{ab} is irreducible with respect to satisfying $P(a, x)$ it follows that $x \notin P_{az_i}$ for each i but $x \in \limsup P_{az_i} = P_{ab}$. We may suppose that for each i there is $x_i \in P_{az_i} - P_{az_{i-1}}$ such that $\lim x_i = x$. Since P_{ab} contains at most countably many local cutpoints of X we may suppose each x_i is not a local cutpoint of X .

For each $i = 2, 3, \dots$ there is a continuum $P_{d_{i-1} c_{i+1}} \subset P_{d_{i-1} b} \cap P_{ac_{i+1}}$ such that $P_{d_{i-1} c_{i+1}}$ is irreducible with respect to satisfying $P(d_{i-1}, c_{i+1})$. Notice that $x_i, z_i \in P_{d_{i-1} c_{i+1}}$.

For each i let W_i be a neighbourhood of $P_{ab} - K_i$ and A_i a finite set such that A_i separates a and b in W_i and $W_{i+j} \subset W_i \cup U_i$ for all $i, j = 1, 2, \dots$. Since x_i and z_i are not local cutpoints of X we may suppose that $x_i, z_i \notin A_j$ for all i and j . Note that for $j = 1, 2, \dots, z_{i+j}$ and x_{i+j} lie in the component of $W_i - A_i$

which contains b and, for $j = 1, \dots, n-1$, z_{i-j} and x_{i-j} lie in the component of $W_i - A_i$ which contains a .

For each $i = 2, 4, 6, \dots$, $P_{d_{i-1}c_{i+1}} \subset W_1 \cap \dots \cap W_i \cap W_{i+1}$ so there exists an arc C_i joining x_i to z_i in $(W_1 \cap \dots \cap W_{i-1} \cap W_{i+1}) - (A_1 \cup \dots \cup A_{i-1} \cup A_{i+1})$. The arcs C_i are pairwise disjoint by construction. This contradicts our assumption that every sequence of disjoint subcontinua of X is a null sequence. The claim is proved.

CLAIM 5. *If $x \in P_{ab}$ then there is a unique continuum P_{ax} (resp. P_{xb}) in P_{ab} which is irreducible with respect to satisfying $P(a, x)$ (resp. $P(x, b)$).*

PROOF. Just suppose A and B are two subsets of P_{ab} which are irreducible with respect to satisfying $P(a, x)$. Let $y \in A - B$ and let $z \in B - A$ such that y and z are not local cutpoints of X . Let $P_{yb} \subset P_{ab}$ be a continuum which is irreducible with respect to satisfying $P(y, b)$. By the remark following Claim 3 $P_{ab} = A \cup P_{yb}$ thus $z \in P_{yb}$. By Claim 4 there exists a continuum $P_{zb} \subset P_{yb} - \{y\}$ such that P_{zb} satisfies $P(z, b)$. Thus, $B \cup P_{zb}$ is a proper subcontinuum of P_{ab} which satisfies $P(a, b)$. This is a contradiction. The claim is proved.

CLAIM 6. *If $x, y \in P_{ab}$ then either $P_{ax} \subset P_{ay}$ or $P_{ay} \subset P_{ax}$.*

PROOF. Suppose $P_{ax} \not\subset P_{ay}$ and $P_{ay} \not\subset P_{ax}$. By Claims 2 and 5, $x \notin P_{ay}$ and $y \notin P_{ax}$. Since P_{ab} contains at most countably many local cutpoints of X , we may suppose x and y are not local cutpoints of X .

Since $P_{ax} \cup P_{xb} = P_{ab}$, $y \in P_{xb}$. By Claim 4, $P_{yb} \not\subset P_{xb}$. Thus, $x \notin P_{yb}$ and $P_{ay} \cup P_{yb}$ is a proper subcontinuum of P_{ab} which satisfies $P(a, b)$. This is a contradiction. The claim is proved.

Let $2^{P_{ab}}$ denote the space of closed subsets of P_{ab} with the Hausdorff metric topology. Let C denote the closure in $2^{P_{ab}}$ of $\{P_{ax} | x \in P_{ab}\}$.

CLAIM 7. *C is homeomorphic to the closed unit interval $[0, 1]$.*

PROOF. $2^{P_{ab}}$ is a compact metric space that is partially ordered by inclusion. This partial order is a closed relation on $2^{P_{ab}}$. Since C is compact, we need only prove that C is connected and totally ordered under inclusion (Ward [9]).

Let $C \in C$. We start by showing that if $x \in C$ then $P_{ax} \subset C$. For let x_i be a sequence in P_{ab} such that P_{ax_i} converges to C in C . If, for arbitrarily large i , $x \in P_{ax_i}$ then $P_{ax} \subset P_{ax_i}$ for all such i by Claims 2 and 5 and hence $P_{ax} \subset \limsup P_{ax_i} = C$. If, on the other hand, $P_{ax_i} \subset P_{ax}$ for arbitrarily large i and if U is a neighbourhood of C such that a and x can be separated by a finite

set in U , then for some sufficiently large i P_{ax_i} can be separated in U by that same finite set. This contradicts Claim 2. By Claim 6, these are the only two cases we need consider. Thus, $x \in C$ implies $P_{ax} \subset C$.

Let $C, D \in \mathcal{C}$ such that $C \not\subset D$. Let $x \in C - D$ such that x is not a local cutpoint of X and let $y \in D$. Then $x \notin P_{ay}$ since $P_{ay} \subset D$. By Claim 6 $y \in P_{ax} \subset D$. Thus $D \subset C$ and $\mathcal{C}$ is totally ordered by inclusion.

If C is not connected then there exist $C, D \in \mathcal{C}$ with $C \subsetneq D$ such that for each $E \in \mathcal{C}$ either $E \subset C$ or $D \subset E$. Now, $D - C$ contains at least two points x and y such that x and y are not local cutpoints of X . Then $P_{ax} = P_{ay} = D$. By Claim 4 $x = y$. With this contradiction we conclude that $\mathcal{C}$ is connected. The claim is proved.

CLAIM 8. *If (x_i) is a sequence in P_{ab} such that the sequence (P_{ax_i}) converges in $\mathcal{C}$ then (x_i) converges in P_{ab} .*

PROOF. Just suppose that (x_i) and (y_i) are two sequences in P_{ab} which converge to x and y respectively such that the sequences (P_{ax_i}) and (P_{ay_i}) both converge to $C \in \mathcal{C}$. We may suppose that, for each i , x_i and y_i are not local cutpoints of X and neither sequence is constant.

By Claim 6 we need to consider only three cases.

Case 1. For each i , $P_{ax_i} \subset P_{ax_{i+1}}$ and $P_{ay_i} \subset P_{ay_{i+1}}$. If, for each i and j , $P_{ax_i} \subset P_{ay_j}$ then $C = P_{ay_1}$ and the sequence (y_i) would have to be constant. We may suppose, therefore, that for each i $P_{ax_i} \subset P_{ay_i} \subset P_{ax_{i+1}}$. As in the proof of Claim 4 if $y \in P_{ax_i} \cap P_{x_{i+1}b}$ and y is not a local cutpoint of X then $y = x_i$. Since P_{ab} contains at most countably many local cutpoints of X , $P_{ax_i} \cap P_{x_{i+1}b}$ is at most a countable set. Since $P_{ay_{i-1}} \not\subset P_{ax_i}$ it follows that $P_{ab} - (P_{x_{i+1}b} \cup P_{ay_{i-1}})$ is a nonempty open subset of P_{ab} which is contained in P_{ax_i} . As in Claim 4 there exists an arc I_i of local cutpoints of P_{ab} such that $I_i \subset P_{ab} - (P_{x_{i+1}b} \cup P_{ay_{i-1}}) \subset P_{ax_i} - P_{ay_{i-1}}$. Now construct as in Claim 4 a sequence (A_i) of pairwise disjoint arcs such that for each i $x_i, y_i \in A_i$. Since every sequence of disjoint continua in X is null, $x = y$.

Case 2. For each i $P_{ax_i} \supset P_{ax_{i+1}}$ and $P_{ay_i} \supset P_{ay_{i+1}}$.

Argue as in Case 1.

Case 3. For each i $P_{ax_i} \subset P_{ax_{i+1}}$ and $P_{ay_i} \supset P_{ay_{i+1}}$. Then $P_{ax_i} \subset P_{ay_j}$ for each i and j . Let U be a neighbourhood of x in P_{ab} such that the boundary of U contains no local cutpoints of X and $y \notin U$. For each i let $z_i \in P_{x_i y_i} \cap (\text{boundary of } U)$. Then, $P_{ax_i} \subset P_{az_i} \subset P_{ay_i}$ so that $\lim P_{az_i} = C$.

If for some subsequence (z_{ij}) $P_{az_{ij}} \subsetneq P_{az_{ij+1}}$ for each j , then we are in Case 1 with the sequences (x_i) and (z_{ij}) . If for some subsequence (z_{ij}) $P_{az_{ij}} \supsetneq P_{az_{ij+1}}$

for each j then we are in Case 2 with the sequences z_{ij} and y_i). If the sequence z_i contains a constant subsequence we may suppose $z_i = z$ for each i . Let w_i be a sequence in P_{az} which converges to z . Then $\lim P_{aw_i} = P_{az} = C$. We may now apply Case 1 to the sequences w_i and x_i .

The claim is now proved.

We are now in a position to adapt the proof of Theorem 6 in [2] to the continuum P_{ab} .

We shall attach to P_{ab} an infinite sequence of pairwise disjoint closed sets A_i such that no pair of points of P_{ab} can be separated by a finite set in $P_{ab} \cup (\bigcup A_i)$.

Let h be a homeomorphism of the closed unit interval $[0, 1]$ onto C such that $h(0) = \{a\}$ and $h(1) = P_{ab}$. Define $f: [0, 1] \rightarrow P_{ab}$ by letting

$$f(r) = \begin{cases} x, & \text{if } h(r) = P_{ax}, \\ \lim x_i, & \text{if } h(r) = \lim P_{ax_i}. \end{cases}$$

By Claim 8 f is a continuous function. Notice that if $x \in P_{ab}$ is not a local cut-point of X then $f^{-1}(x)$ is a singleton.

If A is a set in X and $\epsilon > 0$ we let $S(A, \epsilon)$ denote the ϵ -neighbourhood of A in X .

By the proof of Claim 4 if $r \in [0, 1]$, then $h(r) = \{f(r)\} \cup (\bigcup \{h(s) \mid s < r\})$.

Let $0 < r < 1$. For $\epsilon > 0$ there exists, by the proof of Claim 4, $\delta > 0$ such that $r < s < \delta + r$ implies $h(s) \subset h(r) \cup S(f(r), \epsilon)$.

Let $Y_r = \bigcup \{P_{xb} \mid x \in P_{ab} - h(r)\}$. Let r_i be a sequence in $[0, 1]$ which is strictly decreasing to r such that, for each i , $f(r_i)$ is not a local cutpoint of X . It can be shown as in the proof of Claim 4 that $Y_r = \bigcup P_{f(r_i)b}$ and $Y_r \cup \{f(r)\}$ is compact.

If $0 < s < 1$ and U is any neighbourhood of $h(s)$ then no pair of points of $h(s)$ can be separated in U by a finite set. We may suppose therefore that either

- (1) for each i there exist $c_i, d_i \in [0, 1]$ with $c_i < r < d_i$ such that $f(c_i) = f(d_i) \in (h(r) \cap Y_r \cap S(f(r), 1/i)) - \{f(r)\}$, or
- (*) for each i there exists an arc $C_i \subset S(f(r), 1/i)$ such that $C_i \cap h(1)$ consists precisely of the two endpoints of C_i ,
- (2) one endpoint of C_i is in $h(r) - \{f(r)\}$ and the other is in $h(1) - h(r)$. The arcs C_i may be taken to be pairwise disjoint.

Suppose (1) holds. Let $E_i = \{f(c_i)\}$. We wish to show that $\lim c_i = r$. Just suppose that for each i $c_i \leq s < r$. Then $h(s) \cup Y_r$ is a continuum in P_{ab}

which satisfies $P(a, b)$. If $s < t < r$ and $f(t)$ is not a local cutpoint of X then $f(t) \notin h(s) \cup Y_r$. This contradicts the assumption that P_{ab} is an irreducible continuum with respect to satisfying $P(a, b)$. Thus, $\lim c_i = r$. Similarly, $\lim d_i = r$.

Now suppose (2) holds. Since every sequence of disjoint subcontinua of X is a null sequence, there exists a sequence ϵ_i of positive numbers converging to zero such that if D_i is the component of $[S(h(1), \epsilon_i) \cup C_i] - h(1)$ which meets C_i then $D_i \cap D_j = \emptyset$ for $i \neq j$ and the diameters of the D_i converge to 0.

Let $M_i = \text{Cl}(D_i) \cap (h(r) - \{f(r)\})$ and $N_i = \text{Cl}(D_i) \cap (Y_r - \{f(r)\})$. If M_i (resp. N_i) has an isolated point let c_i (resp. d_i) be an isolated point of M_i (resp. N_i). If M_i (resp. N_i) has no isolated points let $c_i = \inf \{s \in [0, 1] \mid \text{Cl}(D_i) \cap h(s) \text{ is uncountable}\}$ (resp. $d_i = \sup \{s \in [0, 1] \mid \text{Cl}(D_i) \cap P_{f(s)b} \text{ is uncountable}\}$). Then $h(c_i) \cap \text{Cl}(D_i)$ is at most countable.

As in (1) $\lim c_i = \lim d_i = r$. Let E_i be an arc in $D_i \cup \{c_i, d_i\}$ with endpoints c_i and d_i .

We wish to prove that for each $\epsilon > 0$ and each $s \in [0, 1]$ either

$$(**) \quad (h(s) \cap Y_s \cap S(f(s), \epsilon)) - \{f(s)\} \neq \emptyset$$

or there exists an arc in $S(f(s), \epsilon) - E_i$ which joins $h(s)$ to $h(1) - h(s)$. We need only consider $s \in [0, 1]$ such that $f(s) = f(c_i)$ or $f(s) = f(d_i)$.

Clearly, (**) is satisfied if both M_i and N_i have an isolated point. Suppose, therefore, that M_i does not have an isolated point. Let $s \in [0, 1]$ such that $f(s) = f(c_i)$ and suppose $\epsilon > 0$ is given such that

$$(h(s) \cap Y_s \cap S(f(s), \epsilon)) - \{f(s)\} = \emptyset.$$

By (*) there is a sequence of arcs $F_j \subset S(f(s), \epsilon) - \{f(s)\}$ which join $h(s)$ to $h(1) - h(s)$ such that $\lim F_j = \{f(s)\}$. Let e_j and f_j be the endpoints of F_j . We may suppose $F_j \cap h(1) = \{e_j, f_j\}$ where $e_j \in h(s)$ and $f_j \in h(1) - h(s)$.

Just suppose that for each j $E_i \cap F_j \neq \emptyset$. Then each $e_j \in \text{Cl}(D_i)$. By the choice of c_i and by the assumption that M_i is a perfect set each neighbourhood of e_j contains uncountably many points of Y_{c_i} . Since $Y_{c_i} \cup \{f(c_i)\}$ is compact $e_j \in Y_{c_i}$. If $s \leq c_i$ then

$$e_j \in (h(s) \cap Y_s \cap S(f(s), \epsilon)) - \{f(s)\}$$

which is a contradiction. If $c_i < s$ then for each j let $e'_j \in [0, 1]$ such that $f(e'_j) = e_j$. We get as in (1) that $\lim e'_j = s$ and hence eventually $e_j \in Y_s - h(c_i)$ which is again a contradiction. We conclude that for all sufficiently large j $E_i \cap F_j = \emptyset$. Thus, (**) is satisfied.

If there exist $c < \frac{1}{2} < d$ such that $f(c) = f(d) \in S(f(\frac{1}{2}), 1) - \{f(\frac{1}{2})\}$ let $C(\frac{1}{2}, 1) = \{f(c)\}$. Otherwise, let $C(\frac{1}{2}, 1)$ be an arc in $S(h(1), 1)$ with endpoints

$f(c)$ and $f(d)$ where $\frac{1}{4} < c < \frac{1}{2} < d < \frac{3}{4}$ and $C(\frac{1}{2}, 1)$ is obtained as was E_i above.

Let $A_1 = C(\frac{1}{2}, 1)$. Suppose $A_1, \dots, A_{n-1}$ have been constructed to be pairwise disjoint closed sets such that for each $i = 1, \dots, n-1$

(i) A_i is the union of a finite number of arcs and points each of which was obtained as was E_i above,

(ii) for each $a \in [1/2^i, 1 - 1/2^i]$ there is a component C of A_i such that C meets both $f([1/2^i, a]) - \{f(a)\}$ and $f([a, 1 - 1/2^i]) - \{f(a)\}$,

(iii) $P_{ab} \cap A_i \supset P_{ab} - f([0, 1/2^{i+1}] \cup [1 - 1/2^{i+2}, 1])$.

For each $x \in [1/2^n, 1 - 1/2^n]$ let $C(x, n) \subset S(h(1), 1/2^{n-1}) - (A_1 \cup \dots \cup A_{n-1})$ be an (possibly degenerate) arc chosen as was E_i with endpoints $f(c(x, n))$ and $f(d(x, n))$ where $1/2^{n+1} < c(x, n) < x < d(x, n) < 1 - 1/2^{n+1}$, $C(x, n)$ minus its endpoints lies in $X - h(1)$ and $C(x, n)$ satisfies (**). The set of open intervals $]c(x, n), d(x, n)[$ such that $x \in [1/2^n, 1 - 1/2^n]$ is an open cover for the compact set $[1/2^n, 1 - 1/2^n]$ hence there exists a minimal finite set $\{x_1, \dots, x_k\} \subset [1/2^n, 1 - 1/2^n]$ such that

$$]c(x_1, n), d(x_1, n)[\cup \dots \cup]c(x_k, n), d(x_k, n)[$$

covers $[1/2^n, 1 - 1/2^n]$. It is clear that conditions (i)–(iii) are satisfied for n .

Let $A_n = C(x_1, n) \cup \dots \cup C(x_k, n)$.

Let $Y = P_{ab} \cup A_1 \cup A_2 \cup \dots$. Since every sequence of disjoint subcontinua of X is a null sequence Y is a continuum.

To prove that Y contains a connected set that is not arcwise connected it suffices by Theorem 1 and the fact that P_{ab} contains only countably many local cutpoints of X to prove that if z is a local cutpoint of Y then either $z \in A_i$ for some i or z is a local cutpoint of X . Let $z \in P_{ab}$ such that z is not a local cutpoint of X and $z \notin A_i$ for any i . It follows from the fact that $f^{-1}(z)$ is a singleton that if U is any connected neighbourhood of z in P_{ab} such that $U - \{z\}$ has more than one component then $U - \{z\}$ has exactly two components one of which is contained in $\{f(y) | y < f^{-1}(z)\}$ and the other is in $\{f(y) | f^{-1}(z) < y\}$. It is now easy to see that z is not a local cutpoint of Y and the theorem is proved.

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